Riding the Waves

McCormick and Kraemer describe an analytical method for calculating hydrodynamic coefficients for a vertical cylinder.

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Who Should Read This Paper?

The heaving motions of a vertical, circular cylinder are of interest in many areas of ocean engineering. The primary application of the research reported here is in the design of spar buoys and tension leg platforms. The paper describes a spreadsheet friendly analytical method of determining hydrodynamic coefficients. Readers will learn about an alternative approach to the computation of the added-mass (the ambient water mass excited by the motions of the body) and radiation damping (the damping due to the energy lost to the waves created by the body motions) of a heaving, circular cylinder.

Why it is Important

Existing methods of determining hydrodynamic coefficients for a vertical cylinder are primarily numerical in nature. These methods are time-consuming and computationally intensive. The analytical method described in this paper can provide good results in a relatively short period of time. Engineers in the offshore oil industry could use the method in the conceptual design phase of platform design.

About the Authors

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LONG-WAVE APPROXIMATIONS OF THE ADDED-MASS AND RADIATION DAMPING OF A HEAVING, VERTICAL, CIRCULAR, CYLINDRICAL IN WATERS OF FINITE DEPTH

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ABSTRACT

Quasi closed-form expressions for the added-mass and radiation damping coefficients for a heaving vertical, truncated, circular cylinder acting in waters of finite depth are derived. The expressions are based on the long-wave assumption in that the effects of the high-frequency evanescent waves are considered to be of second-order and the wave number-radius (*ka*) range is less than 0.3. The expressions are found to be both utilitarian and relatively accurate. The latter conclusion is based on comparisons of results with those obtained using the Yeung [1981] analysis and experimental results. The comparisons show that the analysis is conservative in the prediction of the added-mass coefficient, but agrees quite well with the experimentally determined radiation damping coefficient values. The Yeung analysis under-predicts the added-mass coefficient, but predicts the radiation damping well. Results of both analyses compare rather well with the experimental results.

1. INTRODUCTION

Of all of the structures used in ocean engineering applications, the vertical, circular cylinder is the most popular. For example, the geometry is that of the can-buoy which is found in all navigable waters. As a result, accurate analyses of the exciting forces, reaction forces and motions for the structure in waves have been the goals of ocean engineers for more than a century. In addition to can buoys, this simple structure is also the basic structural segment of many tension-leg platforms (TLP), semi-submersibles and wave-energy conversion systems. For these floating structures, the moving bodies act as wave-makers in all of the modes of motion. Of particular interest herein are the added-mass and radiation damping of the vertical truncated cylinder experiencing pure heaving.

There are a number of analytical studies of the hydrodynamic coefficients of a heaving, vertical, circular cylinder. In addition to that of Yeung [1981], some of these are contained in the papers of McCormick [1983], Huse [1990], Huse and Utnes [1994], O'Kane *et al*. [2002] and Tao and Cai [2004]. All of these are secondary efforts. That is, the main focus of these papers is on either wave-energy conversion or the motions of a tension-leg platform.

The purpose of this study is to provide expressions for the added-mass and radiation damping coefficients for a heaving cylinder which are both easy to apply and accurate. The approach taken here is to establish expressions for the heaving potential function, assuming that the effects of the evanescent waves created by the motions are negligible. Physically, neglecting the evanescent waves is justified by assuming that the waves created are relatively long. That is, the wave number-radius product (*ka*) is relatively small. The approach taken in the analysis is based on both the conservation of mass and the conservation of the energy-flux. This approach is different than that of the classical analysis of Yeung [1981]. Yeung treats the equation of continuity in the form of a radial Laplace equation as both a homogeneous differential equation and an inhomogeneous differential equation and obtains physically plausible results. One of the solutions used by Yeung is a polynomial expression that satisfies the radial equation. In the present study, a circular-function solution of the radial equation is used.

The added-mass and radiation damping coefficient expressions derived in this paper are applied to experimental data obtained at the U. S. Naval Academy.

2. ANALYSIS

(a) Velocity Potentials

Consider the vertical, truncated, circular cylinder of radius a and draft d sketched in Figure 1. This float is supported by a vertical rigid staff to insure pure heaving motions. These motions are somehow excited sinusoidally. The forced heaving motions of the body can then be represented by

$$Z(t) = Z_0 e^{-i\omega t} \tag{1}$$

where Z_0 is the complex amplitude and ω is the circular excitation frequency, $2\pi / T$, where T is the period of motion. In Figure 1, the oscillating vertical velocity vz is the first time-derivative of $Z(t)$. The fluid motions associated with the structural motion, assuming irrotationality, are represented by a velocity potential (Φ) that satisfies Laplace's equation in cylindrical coordinates. Subject to the seafloor condition $\partial\Phi/\partial z\big|_{z=-h} = 0$, the near-field (r > a) potential is assumed to have the form

$$\Phi = A\cosh\left[K(z+h)\right]B_n(kr)\cos(n\beta)e^{-i\omega t} \tag{2}$$

where the coefficient A is to be determined. Referring to Figure 2, the coefficient K depends on the fluid region under consideration. Those regions are Region 1, where $r \le a$ and $-d > z > -h$, and Region 2, where $r > a$ and $0 > z > -h$. Also in equation (2) is $B_n(kr)$, a Bessel function of some kind and of integer-order n. Note that the form of the expression in equation (2) satisfies Laplace's condition for any finite value of the constant.

It should be mentioned here that in wave-maker theory, the analyst normally assumes that the potential is composed of a single traveling wave plus an infinite number of evanescent standing waves (see, for example, Dean and Dalrymple [1984]). The potential in equation (2) is, then, a simplification of the wave-maker theory.

As stated earlier, the method used herein of determining the added-mass and radiation damping coefficients differs somewhat from Yeung [1981] and others. Referring again to the sketch in Figure 2, the fluid volume is composed of two regions, as is done by Yeung [1981] and others. Region 1 is directly under the body while Region 2 is external to Region 1. The water particle motions in both regions contribute to both the inertial reactions on the body. The radiation damping, however, depends on the particle motions in Region 2 excited by the motions in Region 1.

The potential in Region 2 is assumed to have a form similar to that in equation (2) where the Bessel function is the Hankel function of the first kind. Since the radiating wave pattern is symmetric about the z-axis, there is no β-dependence. That is, $n = 0$ in equation (2). To satisfy the free-surface and seafloor conditions, K must be the wave number $k = 2\pi/\lambda$, where λ is the wavelength of the generated waves. The assumed potential in Region 2 is, then,

$$\Phi_2 = A_2 \cosh\left[K(z+h)\right] H_0^1(kr) e^{-i\omega t} \tag{3}$$

where the subscript "2" refers to the region. The alternating volume efflux and influx through the circular cylindrical boundary separating Regions 1 and 2 must be equal to the alternating influx and efflux caused by the heaving motions. This allows us to write

$$\begin{aligned} V_z \pi a^2 &= -i\omega Z_0 \pi a^2 e^{-i\omega t} \\ &= \int_0^{2\pi} \int_{-h}^{-d} \frac{\partial \Phi_2}{\partial r}\bigg|_{r=a} a\,dz\,d\beta = A_2 \frac{2\pi a}{k} \left\{\sinh\left[k(h-d)\right]\right\} H_0^{(1)\prime}(ka) e^{-i\omega t} \end{aligned} \tag{4}$$

Solving for A_2, we find that the expression for the exterior potential is

$$\Phi_2 = i\omega \frac{Z_0}{2} a \left\{ \frac{\cosh\left[k(z+h)\right]}{\sinh\left[k(h-d)\right]} \right\} \frac{H_0^{(1)}(kr)}{H_1^{(1)}(ka)} e^{-i\omega t} \tag{5}$$

where $H_0^{(1)\prime}(ka) = -kH_1^{(1)}(ka)$ has units of 1/length. Abramowitz and Stegun [1965] discuss the various Bessel functions.

The solution of Laplace's equation in Region 1 must be both finite at $r = 0$ and symmetric about the z axis. In addition, it must satisfy the seafloor condition and the condition on the bottom of the cylinder. These requirements and Laplace's equation could potentially be satisfied by

$$\Phi_1 = \left\{A_1 \cos\left[k(z+h)\right] I_0(kr) + B_1\right\}^{-i\omega t} \tag{6}$$

Here, the coefficient K in equation (2) is ik. The constant B_1 is a mating constant needed to couple the potential functions in Regions 1 and 2. The function $I_0(kr)$ is a modified Bessel function of the first kind, order 0. Using the velocity potential in equation (6), we find that the vertical velocity spatially averaged over the bottom of the cylinder is

$$-i\omega Z_0 e^{-i\omega t} = -\frac{1}{\pi a^2} \int_0^{2\pi} \int_0^{a} \frac{\partial \Phi_1}{\partial z}\bigg|_{z=-d} r\,dr\,d\beta \tag{7}$$

From this expression, the constant A_1 is found. We also require that the total energy flux across the boundary separating Regions 1 and 2 be conserved. Mathematically, this assumption is

$$\rho \int_0^{2\pi} \int_{-h}^{-d} \frac{\partial \Phi_1}{\partial r}\bigg|_{r=a} \frac{\partial \Phi_1}{\partial t}\bigg|_{r=a} adzd\beta$$
$$= \rho \int_0^{2\pi} \int_{-h}^{-d} \frac{\partial \Phi_2}{\partial r}\bigg|_{r=a} \frac{\partial \Phi_2}{\partial t}\bigg|_{r=a} adzd\beta \qquad (8)$$

The resulting velocity potential expression is found to be

$$\Phi_1 = i\omega \frac{Z_0}{2} a \left\{ \begin{array}{l} \dfrac{I_0(kr)\cos[k(z+h)]}{I_1(ka)\sin[k(h-d)]} \\ -\dfrac{I_0(ka)}{I_1(ka)} \dfrac{1}{\sin^2[k(h-d)]} \left\langle \begin{array}{c} \dfrac{k(h-d)}{2} \\ +\dfrac{1}{4}\sin[2k(h-d)] \end{array} \right\rangle \\ -\dfrac{H_0^{(1)}(ka)}{H_1^{(1)}(ka)} \dfrac{1}{\sinh^2[k(h-d)]} \left\langle \begin{array}{c} \dfrac{k(h-d)}{2} \\ +\dfrac{1}{4}\sinh[2k(h-d)] \end{array} \right\rangle \end{array} \right\} e^{-i\omega t} \qquad (9)$$

where, $I_0'(ka) = -kI_1(ka)$ from Abramowitz and Stegun [1965].

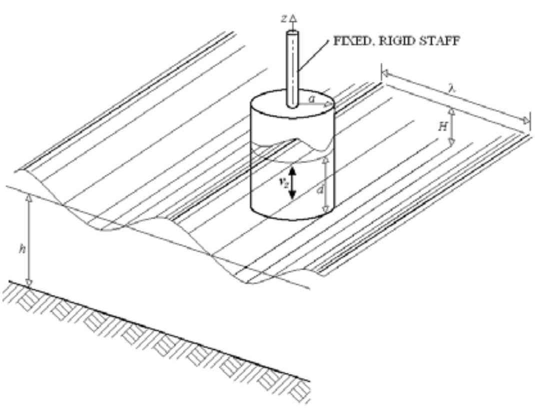

Figure 1. Sketch of a vertical circular cylinder constrained to heave.

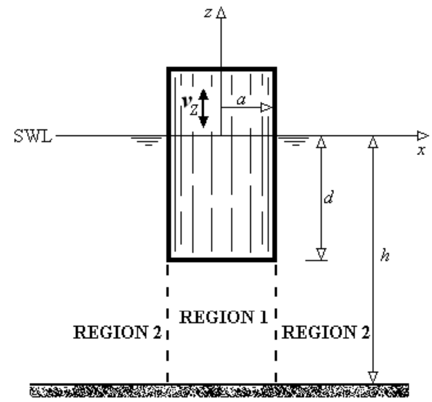

Figure 2. Segregated regions for the heaving analysis.

(b) Reaction Forces

From the linear theory, the hydrodynamic pressure at any point in Region 1 is $p_1 = -\rho \partial\Phi/\partial t$ where ρ is the mass-density of the fluid. The (reaction) force on the bottom of the cylinder resulting from the potential function in equation (9) is

$$F_z(\omega, t) = -i\rho 2\pi\omega \int_0^a \Phi_1 \bigg|_{z-d} rdr = M_z \frac{d^2 z}{dt^2} + R_z \frac{dz}{dt} \qquad (10)$$

where M_z is the added-mass for the heaving cylinder and R_z is the radiation damping coefficient. By combining the expressions in equations (1), (9), and (10), the respective expressions for the added-mass and radiation damping coefficient are

$$M_z = \rho\pi a^3 \left\{ \begin{array}{l} \dfrac{1}{8\sin^2[k(h-d)]} \left\langle \begin{array}{c} \dfrac{k(h-d)}{2} \\ +\dfrac{1}{4}\sin[2k(h-d)] \end{array} \right\rangle \dfrac{I_0(ka)}{I_1(ka)} \\ +\dfrac{1}{8\sinh^2[k(h-d)]} \left\langle \begin{array}{c} \dfrac{k(h-d)}{2} \\ +\dfrac{1}{4}\sinh[2k(h-d)] \end{array} \right\rangle H_{\Re}(ka) \\ -\dfrac{1}{(ka)} \dfrac{\cos[k(h-d)]}{\sin[k(h-d)]} \end{array} \right\} \qquad (11)$$

and

$$R_z = \frac{\rho \pi a^3 \omega}{8} \frac{1}{\sinh^2[k(h-d)]} \left\langle \begin{matrix} 2k(h-d) \\ + \sinh[2k(h-d)] \end{matrix} \right\rangle H_{\Re}(ka) \quad (12)$$

In these equations, the following notation is used.

$$\begin{aligned} \frac{H_0^{(1)}(ka)}{H_1^{(1)}(ka)} &= \frac{J_0(ka) + iY_0(ka)}{J_1(ka) + iY_1(ka)} \\ &= \frac{J_0(ka)J_1(ka) + Y_0(ka)Y_1(ka)}{J_1^2(ka) + Y_1^2(ka)} \\ &\quad - i\frac{J_0(ka)Y_1(ka) - Y_0(ka)J_1(ka)}{J_1^2(ka) + Y_1^2(ka)} \\ &\equiv H_{\Re}(ka) + iH_{\Re}(ka) \end{aligned} \quad (13)$$

(c) Yeung's Long-Wave Approximations

The elegant analysis of Yeung [1981] is used as a standard for the present study. Yeung essentially extends the analysis of Garrett [1971] to a vertical, truncated, circular cylinder undergoing surging, heaving and pitching motions in the *x-z* plane. The Garrett analysis is directed at the wave-induced forces on the cylinder where the cylinder is fixed in place. The analyses of both Garrett [1971] and Yeung [1981] are expansion types where evanescent waves are present. As stated earlier in the paper, our interest is in obtaining simplified expressions for the added-mass and radiation damping coefficient that apply in long waves. That is where *ka* and *kh* are relatively small. Under the long-wave assumption, the evanescent terms in Yeung's analysis are considered to be of second order. The long-wave approximations of the added-mass and radiation damping are

$$M_z \cong \frac{\rho \pi a^3}{2}\left[\frac{a}{4(h-d)} + \frac{1}{kh}H_{\Re}(ka)\right] \quad (14)$$

and

$$R_z = \frac{\rho \pi a^3}{2}\frac{\omega}{kh}H_{\Re}(ka) \quad (15)$$

One apparent difference in the damping coefficient expressions in equations (12) and (15) is that the expression in the former depends on the bottom clearance (h - d) while the latter does not.

To determine the accuracy of the expressions in equations (11) and (12), experimental results obtained in the U. S. Naval Academy's Hydromechanics Laboratory are presented in the next section. The experimental setup is described by McCormick, Coffee and Richardson [1982].

3. EXPERIMENTAL STUDY

The experiment described here was conducted in the 37m long wave and towing tank at the U.S. Naval Academy. The width of the tank is 2.44m. Referring to the schematic in Figure 1, the diameter of the cylinder was $D = 2a = 0.218$m. The heaving motions of the body were measured using a sonic gauge mounted over the top of the cylinder and affixed to a towing carriage. Two draft conditions were used, those being $d = 0.914$m and 1.07m. For each draft condition, there were seven water-depth values. Those were $h =$ 1.22m, 1.30m, 1.37m, 1.45m, 1.52m, 1.60m, and 1.68m.

To determine the added-mass (M_Z) and the total damping (R_Z) for each depth condition, the cylinder was displaced 0.0762m and released, and the subsequent decaying heaving motions were measured. The results obtained for the 0.914m draft are presented in Table 1; while, those obtained for the 1.070m draft are in Table 2. In the tables, ω is the damped natural circular frequency and R_{cr} is the critical damping, defined as

$$R_{cr} = 2\sqrt{\rho g \pi a^2 (m + M_z)} \tag{16}$$

In this expression, m is the displaced water mass. Comparing the values in the last two columns of Tables 1 and 2, the reader can see that the motions were under-damped for both draft values.

4. RESULTS

The non-dimensional added-mass is presented as a function of ka in Figures 3 and 4 for a water depth of 1.22m. The respective draft values for these figures are 0.914m and 1.067m. The non-dimensional damping coefficients are presented as a function of ka in Figures 5 and 6 for the respective conditions in Figures 3 and 4. The experimental data (■) are total damping values, since the radiated wave properties were not measured. That is, the data represent the sum of the radiation damping coefficient and the viscous damping coefficient. We obtain the radiation damping component (▲) of the total damping by

$$\mathrm{R}_z \cong R_Z + R_{Zv} = R_Z + \frac{8}{6}\rho C_d \omega Z a^2 \tag{17}$$

The third and last term of this equation is the time-averaged, equivalent viscous-pressure damping coefficient. The drag coefficient, C_d, is assumed to have a value of approximately 0.8 from the results presented by Hoerner [1965]. To obtain values for the equivalent damping coefficient, the initial amplitude value of 0.0762m was used, as stated in the discussion of Tables 1 and 2. The values of the radiation damping coefficient are, then, obtained from $R_Z = \mathrm{R}_Z - R_{Zv}$.

h (m)	M_z (kg)	ω (rad/s)	R_{cr} (N-s/m)	R_z (N-s/m)
1.22	4.164	3.128	133.6	4.516
1.30	4.164	3.111	133.6	4.543
1.37	3.515	3.142	132.5	4.398
1.45	3.311	3.163	132.1	4.386
1.52	3.311	3.162	132.1	4.122
1.60	3.515	3.132	132.5	4.266
1.68	4.164	3.138	133.6	4.904
averages	3.735	3.139	132.8	4.448

Table 1. Results obtained for draft $d = 0.9144$.

h (m)	M_Z(kg)	ω (rad/s)	R_{cr}(N-s/m)	R_Z(N-s/m)
1.22	3.863	2.920	142.7	4.995
1.30	5.177	2.910	144.8	4.780
1.37	3.863	2.909	142.7	4.767
1.45	3.863	2.952	142.7	4.852
1.52	3.226	2.922	141.7	4.647
1.60	3.544	2.930	142.2	4.380
1.68	3.863	2.921	142.7	4.581
averages	3.914	2.923	142.9	4.715

Table 2. Results obtained for draft $d = 1.067$m.

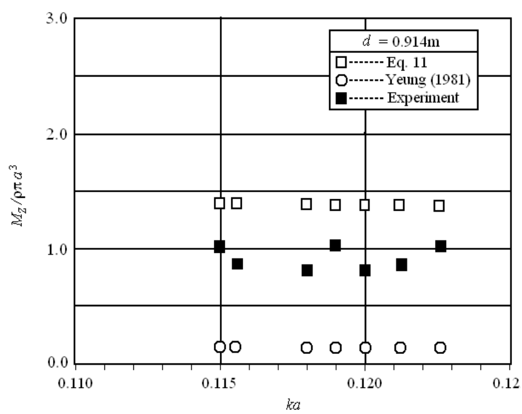

Figure 3. Theoretical and experimental added-mass as a function of ka for a 0.914m draft in 1.22m of water.

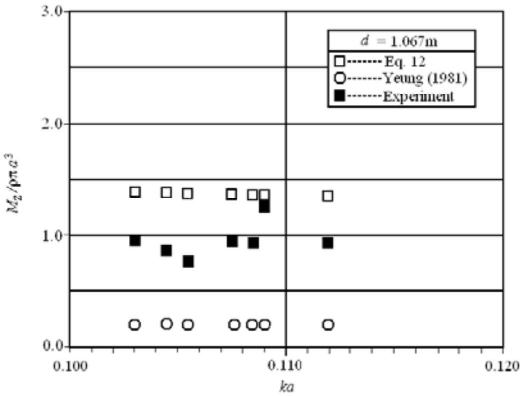

Figure 4. Theoretical and experimental added-mass as a function of *ka* for a 1.067m draft in 1.22m of water.

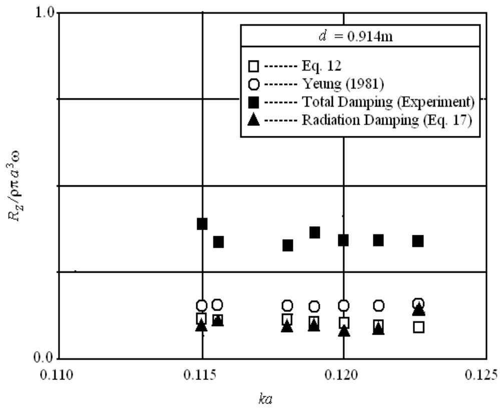

Figure 5. Theoretical and experimental damping coefficients as a function of *ka* for a 0.914m draft in 1.22m of water.

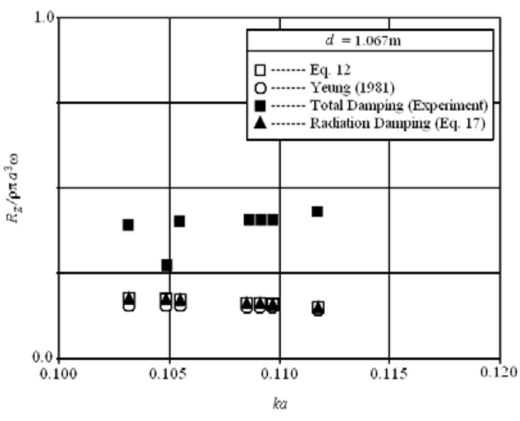

Figure 6. Theoretical and experimental damping coefficients as a function of *ka* for a 0.914m draft in 1.22m of water.

5. DISCUSSION AND CONCLUSIONS

In this study, a trigonometric solution of Laplace's equation is used in the long-wave analysis of the reaction forces on the bottom of a heaving, vertical circular cylinder. This form of the solution is an alternative to that of Yeung [1981] which is a polynomial solution. Comparing results obtained from each analytical method with experimental data, we find that the analysis presented herein over-predicts the added-mass by about 40%. The Yeung analysis under-predicts the added-mass by approximately 300%. These numbers should be taken with "a grain of salt" since the differences in the analytical and experimental values are approximately the same in the non-dimensional curves in Figures 3 and 4.

The damping results in Figures 5 and 6 show that both equation (12) and the Yeung [1981] long-wave expression in equation (15) predict well the radiation damping coefficient values. The differences in the predicted and experimental values increase with the water depth. There is little change in the experimental values with the draft of the cylinder. One might note that the cylinder draft (*d*) appears in equation (12) but not in equation (15).

In order to determine the effects of changing the draft *d* and the bottom clearance *h - d*, we compare the averaged values on the bottom lines of Tables 1 and 2. The added-mass is seen to increase with the draft. As expected, this increase corresponds to a decrease in the damped natural frequency and an increase in the critical damping. The total damping increases with the draft of the cylinder. One would expect the radiation damping to decrease with increasing depth. The increase is assumed to be due to the reduced clearance ($h - d$).

From our observations, we conclude the following. First, the added-mass values predicted by the expression derived herein, equation (11), are conservative but approximates the experimental data well. The values obtained using the Yeung expression in equation (14) do not agree well with the experimental data. Radiation damping values from equation (12) and the Yeung-expression in equation (14) are in excellent agreement with the experimental values. From these observations, then, equation (11) is recommended for the prediction of the added-mass of a heaving cylinder in long waves, and equation (12) is recommended for the prediction of the radiation damping coefficient since both include the draft.

6. REFERENCES

Abramowitz, M. and Stegun, I. 1965 *Handbook of mathematical functions*, Washington, DC: U. S. Government Printing Office.

Dean, R. G. and Dalrymple, R. A. 1984 *Water wave mechanics for engineers and scientists*, Englewood Cliffs, NJ: Prentice-Hall.

Garrett, C. J. R. 1971 Wave forces on a circular dock. *Journal of Fluid Mechanics*, 46, Part I, 129-139.

Hoerner, S. F. 1965 *Fluid-dynamic drag*, published by the author, 2 King Lane, Greenbriar Brick Town, NJ.

Huse, E. 1990 Resonant heave damping of tension leg platforms. In *Proceedings, 22nd Offshore Technology Conf., Houston, TX, May, OTC Paper 6317*, 431-436.

Huse, E. and Utnes T. 1994 Springing damping of tension leg platforms. In *Proceedings, 26th Offshore Technology Conf., Houston, TX, May, OTC Paper 7446*, 259-267.

McCormick, M. E. 1983 Analysis of optimal ocean wave energy conversion. *Jl. Waterway,Port, Coastal and Ocean Engineering (ASCE),* 109 (2), 180-198.

McCormick, M. E., Coffey, J. P. and Richardson, J. B. 1982 An experimental study of wave power conversion by a heaving, vertical, circular cylinder in restricted waters. *Applied Ocean Research*, 4 (2), CML Publications, 107-112.

O'Kane, J. J., Troesch, A. W. and Thiagaragan, K. P. 2002 Hull component interaction and scaling for TLP hydrodynamic coefficients. *Ocean Engineering*, 29 (5) Elsevier Science Ltd., 513-532.

Tao, L. and Cai, S. 2004 Heave motion suppression of a spar with a heave plate. *Ocean Engineering*, 31 (5-6) Elsevier Science Ltd., 669-692.

Yeung, R. W. 1981 Added-mass and damping of a vertical cylinder in finite-depth waters. *Applied Ocean Research*, 3 (3) CML Publications, 119-133.